Reflecting on truth in a partial setting

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Bristol-München Conference on Truth and Rationality
10. 6. 2016

Overview

- The study of reflection principles are important in the arithmetical setting.
- Also for theories of truth the investigation of reflection principles is important and fruitful.
- What about reflection principles in a partial setting?
- What about the connection between reflection and PKF?

Content

Kripke models

- Kripke: Fixed-point construction for different evaluation schemes e.
- monotone operators Γ_e.
- Fixed-points $\Gamma_e(S) = S$ for $S \subseteq \mathbb{N}$.
- Focus: strong Kleene, $e = sk$.
- The minimal fixed-point for strong Kleene I_{sk}.

Axiomatizing Kripke

Axiomatizations:

- KF (Feferman)
  The problem of external and internal logic.
- IKF (Reinhardt) ($\{A \in \mathcal{L}_T \mid \mathrm{KF} \vdash T(\ulcorner A \urcorner)\}$)
  The problem of natural axiomatization.
- PKF (Halbach/Horsten)

In what sense are these axiomatizations and which one is preferable?

ℕ-Categoricity

Suggestion: ℕ-categoricity.
Fix the interpretation of the arithmetical part with the standard model $\mathcal{N}$.
Σ is ℕ-categorical for a set of models M iff

$$(\mathcal{N}, S) \models \Sigma \Leftrightarrow S \in M$$

For the minimal fixed-point:

$$(\mathcal{N}, S) \models \Sigma \Leftrightarrow S = I_{sk}$$

For arbitrary fixed-points:

$$(\mathcal{N}, S) \models \Sigma \Leftrightarrow S = \Gamma_{sk}(S)$$

ℕ-Categoricity

- The minimal fixed-point is Π^1_1-complete (Kripke, Burgess).
- There is no $\mathbb{N}$-categorical axiomatization of the minimal fixed-point.
- KF is an $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points. (Feferman)
- TFB is an $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points. (Leigh)
- IKF is not $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.

Conclusion: KF is at best an axiomatization of arbitrary fixed-points and $\mathbb{N}$-categoricity cannot be the only criterion.

$\mathbb{N}$-Categoricity and partiality

- The set of derivable sequents of PKF is an $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.
- The set of theorems of PKF, i.e. sequents of the form $\Rightarrow A$, is not $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.
- The set of truth sequents $T(\ulcorner A \urcorner) \Rightarrow A$, $A \Rightarrow T(\ulcorner A \urcorner)$ is an $\mathbb{N}$-categorical axiomatization of arbitrary fixed-points.

Infinitary proof systems

Infinitary proof systems allow for characterizations of the minimal fixed-points.

- Cantini has an infinitary proof system (sequent system with ω-rule) characterizing the minimal fixed-point of supervaluation.
- Welch gametheoretic characterization.
- Meadows infinitary tableaux.

Infinitary proof system for strong Kleene

Example SK_∞ a Tait system: Initial sequents

$$\Rightarrow A \text{ (for true atomic arithmetical sentences)}$$

$$\frac{\Rightarrow A}{\Rightarrow \Gamma, T(\ulcorner A \urcorner)} \qquad \frac{\Rightarrow \neg A}{\Rightarrow \Gamma, \neg T(\ulcorner A \urcorner)}$$

$$\omega\text{-rule } \frac{\ldots \qquad A(\underline{n}) \qquad \ldots}{\forall x A(x)} \text{ (for all } n \in \mathbb{N})$$

Then

$$\mathrm{SK}_\infty \vdash A \Leftrightarrow \# A \in I_{sk}$$

Embeddings into infinitary proof systems

Similar to the Gentzen-Schütte method we can look at embeddings into the infinitary proof systems.

- KF cannot be directly embedded.
- An embedding of the theorems of PKF into SK_∞ is possible
  - if PKF $\vdash \Rightarrow A$, then $\# A \in \Gamma_{\omega^\omega}$ (Cantini, Halbach/Horsten).
  - for the language of truth we only have transfinite induction up to ω^ω in PKF.
- IKF is contained in I_{sk}
  - if IKF $\vdash A$, then $\# A \in \Gamma_{\epsilon_0}$ (Cantini).
  - for the language of truth we have transfinite induction up to ϵ_0 in KF.

Uniform reflection as a finitary ω-rule

$$(\mathrm{RFN}^{R}_{\Sigma}) \qquad \frac{\forall x \mathrm{Pr}_{\Sigma}(\ulcorner A\dot{x} \urcorner)}{\forall x A(x)}$$

$$(\mathrm{RFN}_{\Sigma}) \quad \forall x(\mathrm{Pr}_{\Sigma}(\ulcorner A\dot{x} \urcorner) \to A(x)).$$

- Hilbert 1931.
- Shoenfield constructivized version of the ω-rule.
- Feferman 1962 showed the equivalence.

The strength of uniform reflection

For an axiomatizable theory Σ we use $\mathrm{R}(\Sigma) := \mathrm{EA}_T + \mathrm{RFN}_\Sigma$.

- TB_0 is $\mathrm{EA}_T +$ Tarski biconditionals for sentences of $\mathcal{L}_A$.
- UTB_0 is $\mathrm{EA}_T +$ uniform Tarski biconditionals for formulas of $\mathcal{L}_A$.
- TFB_0 is $\mathrm{EA}_T +$ truth and falsity biconditionals for sentences of $\mathcal{L}_P$, i.e. the language of we get by adding F as the dual for T and allow only positive occurrences of T and F.

$$T(\ulcorner A \urcorner) \leftrightarrow A \,\&\, F(\ulcorner A \urcorner) \leftrightarrow \overline{A}$$

- UTFB_0 is $\mathrm{EA}_T +$ uniform truth and falsity biconditionals for formulas of $\mathcal{L}_P$.

Truth and Reflection

- Reflecting on Tarski biconditionals gives uniform Tarski biconditionals.

Lemma (Horsten, Leigh)

$$\mathrm{UTB}_0 \subseteq \mathrm{R}(\mathrm{TB}_0).$$

- Reflecting on typefree truth and falsity biconditionals gives uniform typefree truth and falsity biconditionals.

Lemma (Horsten, Leigh)

$$\mathrm{UTFB}_0 \subseteq \mathrm{R}(\mathrm{TFB}_0).$$

Truth and Reflection

- Reflecting on uniform Tarski biconditionals gives the compositional axioms.

Lemma (Halbach)

$$\mathrm{CT}_0 \subseteq \mathrm{R}(\mathrm{UTB}_0).$$

- Reflecting on uniform truth and falsity biconditionals gives the compositional axioms of KF.

Lemma (Horsten, Leigh)

$$\mathrm{KF} \subseteq \mathrm{R}(\mathrm{UTFB}_0).$$

Partial logic

- The logic is four valued.
- Gaps and gluts.
- Logical consequence for sequents:
  - Truth preservation
  - Falsity antipreservation

Basic

For negation we have contraposition

$$\frac{\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\Gamma}$$

but not

$$\frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \qquad \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$$

We assume as background an arithmetical theory BASIC formulated in $\mathcal{L}_T$: EA_T formulated in a sequent version of partial logic along the lines of Halbach 2014.

Minimal truth TS_0

TS_0 is obtained by extending BASIC with the initial sequents

$T1$	$T(\ulcorner A \urcorner) \Rightarrow A$
$T2$	$A \Rightarrow T(\ulcorner A \urcorner)$

- Simplicity.
- No need for restriction of the language.

Reflection as a rule

Assume some coding of finite sets of formulas $[\Gamma]$, then $[\Gamma\dot{x}]$ denotes the result of substituting in Γ the x-th numeral for x.

$$[\Gamma\dot{x}] \dot{\Rightarrow} [\Delta\dot{x}]$$

denotes the sequent $\Gamma(x) \Rightarrow \Delta(x)$ with the possible free variable x and the dots indicate as usual the use of the sub and num function.
Let Σ be an axiomatizable theory, then $\mathrm{R}(\Sigma) = \mathrm{EA}_T + \mathrm{RFN}^R_\Sigma$.

$$(\mathrm{RFN}^R_\Sigma)\ \frac{\mathrm{Pr}_\Sigma([\Gamma\dot{x}] \dot{\Rightarrow} [\Delta\dot{x}])}{\Gamma(x) \Rightarrow \Delta(x)}$$

From TS_0 to UTS_0

$\mathrm{R}(\mathrm{TS}_0) \vdash$

(i) $A(x) \Rightarrow T(\ulcorner A\dot{x} \urcorner)$;

(ii) $T(\ulcorner A\dot{x} \urcorner) \Rightarrow A(x)$.

Argument: For all formulas $A(x)$ and for all $n \in \mathbb{N}$:

$$\mathrm{TS}_0 \vdash A(\overline{n}) \Rightarrow T(\ulcorner A(\overline{n}) \urcorner).$$

As this is uniform we get in the formalization

$$\mathrm{EA}_T \vdash \Rightarrow \mathrm{Pr}_{\mathrm{TS}_0}([A\dot{x}] \dot{\Rightarrow} [T(\ulcorner A \urcorner)\dot{x}]).$$

With reflection we get

$$\mathrm{R}(\mathrm{TS}_0) \vdash A(x) \Rightarrow T(\ulcorner A\dot{x} \urcorner).$$

Regaining compositional sequents I

$\mathrm{R(TS_0)} \vdash$

(i) $\mathrm{sent}(x), \mathrm{sent}(y), T(x \mathbin{\dot\wedge} y) \Rightarrow T(x) \wedge T(y)$;

(ii) $\mathrm{sent}(x), \mathrm{sent}(y), T(x) \wedge T(y) \Rightarrow T(x \mathbin{\dot\wedge} y)$;

(iii) $\mathrm{sent}(x), \mathrm{sent}(y), T(x \mathbin{\dot\vee} y) \Rightarrow T(x) \vee T(y)$;

(iv) $\mathrm{sent}(x), \mathrm{sent}(y), T(x) \vee T(y) \Rightarrow T(x \mathbin{\dot\vee} y)$;

(v) $\mathrm{sent}(x), \neg T(x) \Rightarrow T(\dot\neg x)$;

(vi) $\mathrm{sent}(x), T(\dot\neg x) \Rightarrow \neg T(x)$.

Regaining compositional sequents II

$\mathrm{R(UTS_0)} \vdash$

(i) $\mathrm{sent}(\underset{.}{\forall}xy), \forall x T(y\dot{x}) \Rightarrow T(\underset{.}{\forall}xy)$;

(ii) $\mathrm{sent}(\underset{.}{\forall}xy), T(\underset{.}{\forall}xy) \Rightarrow \forall x T(y\dot{x})$;

(iii) $\mathrm{sent}(\underset{.}{\exists}xy), \exists x T(y\dot{x}) \Rightarrow T(\underset{.}{\exists}xy)$;

(iv) $\mathrm{sent}(\underset{.}{\exists}xy), T(\underset{.}{\exists}xy) \Rightarrow \exists x T(y\dot{x})$.

Regaining compositional sequents III

$\mathrm{R}(\mathrm{UTS}_0) \vdash$

(i) $\mathrm{ct}(x), T(\mathrm{val}(x)) \Rightarrow T(\underset{\cdot}{T}\, x)$;

(ii) $\mathrm{ct}(x), T(\underset{\cdot}{T}\, x) \Rightarrow T(\mathrm{val}(x))$;

(iii) $\mathrm{ct}(x), \mathrm{ct}(y), \mathrm{val}(x) = \mathrm{val}(y) \Rightarrow T(x \mathbin{\underset{\cdot}{=}} y)$;

(iv) $\mathrm{ct}(x), \mathrm{ct}(y), T(x \mathbin{\underset{\cdot}{=}} y) \Rightarrow \mathrm{val}(x) = \mathrm{val}(y)$.

Observation

$\mathrm{PKF}_0 \subseteq \mathrm{R}(\mathrm{UTS}_0) \subset \mathrm{R}(\mathrm{R}(\mathrm{TS}_0))$

Induction in classical arithmetic

Theorem (Kreisel and Lévy)

$\mathrm{R}(\mathrm{EA}) = \mathrm{PA}$.

Argument for $\supseteq$:
For a formula A with one free variable let $B(x)$ be $A(\overline{0}) \wedge \forall x(A(x) \to A(x+1)) \to A(x)$. Then we can argue in EA by external induction that for all k, $\mathrm{EA} \vdash B(\overline{k})$. Since the size of the proofs can be bound by an elementary function we can formalize the induction in EA. So we get $\mathrm{EA} \vdash \mathrm{Pr}_{\mathrm{EA}}(\ulcorner B\dot{x} \urcorner)$ and with reflection $B(x)$.

Similarly we get $\mathrm{R}(\mathrm{EA}_T) = \mathrm{PA}_T$.

Induction for $\mathcal{L}_T$ (partial)

Instead of using the (schema) of induction, the following rule is adopted:

$$\frac{A(x), \Gamma \Rightarrow \Delta, A(x+1)}{A(\underline{0}), \Gamma \Rightarrow \Delta, A(t)} \text{ (Ind)}$$

In $\mathrm{R}(\mathrm{UTS}_0)$ we get induction for all formulas of $\mathcal{L}_T$ and so

Observation

$\mathrm{PKF} \subseteq \mathrm{R}(\mathrm{UTS}_0)) \subset \mathrm{R}(\mathrm{R}(\mathrm{TS}_0))$.

Transfinite induction

For a fixed ordinal representation, for example with the Cantor normal form for ordinals $< \epsilon_0$ we define:

Definition

Let A be a formula with one free variable

- $Prog(A) := \forall\alpha < \beta A(\alpha) \to A(\beta)$.
- $\mathrm{TI}(A, \beta) := Prog(A) \to \forall\alpha < \beta A(\alpha)$.
- $\mathrm{TI}_{\mathcal{L}}(< \alpha) := \{\mathrm{TI}(A, \beta) \mid A \in \mathcal{L} \,\&\, \beta < \alpha\}$.

Transfinite induction for a language with truth

Lemma

Reflecting on EA_T *gives* $\mathrm{TI}_{\mathcal{L}_T}(<\epsilon_0)$.

Argument: Similar to PA proves transfinite induction up to ϵ_0.
For a formula $A(x)$ define $A'(x)$ to be

$$\forall\beta(\forall\alpha<\beta A(\alpha)\rightarrow\forall\alpha<\beta+\omega^x A(\alpha))$$

Then we show

$$Prog(A)\rightarrow Prog(A').$$

With this

$$\mathrm{TI}(A,\alpha)\Rightarrow\mathrm{TI}(A,\omega^\alpha),$$

and finally

$$\mathrm{TI}_{\mathcal{L}_T}(<\epsilon_0).$$

$\mathrm{TI}_{\mathcal{L}_{\mathcal{T}}}$ in a partial setting

$$Prog(A) := \forall\alpha < \beta A(\alpha) \Rightarrow A(\beta)$$

$$\mathrm{TIR}(A,\beta)\ \frac{Prog(A)}{\Rightarrow \forall\alpha < \beta A(\alpha)}$$

$\mathrm{TIR}_{\mathcal{L}_T}(<\alpha)$ is the closure under the rules $\mathrm{TIR}(A,\beta)$ for all $A \in \mathcal{L}_T$ and for all $\beta < \alpha$.

$\mathrm{TIR}_{\mathcal{L}_T}(< \epsilon_0)$ in R(UTS)?

Basic proof strategy: Show

$$\frac{Prog(A)}{Prog(A')}$$

then closure under $\mathrm{TIR}(A, \beta)$ implies closure under $\mathrm{TIR}(A, \omega^{\beta})$ for all $A \in \mathcal{L}_T$.

Problems for the direct argument

We run into problems if we try to show that

$$\frac{Prog(A)}{Prog(A')}$$

Remember that $A'(x)$ is $\forall\beta(\forall\alpha < \beta A(\alpha) \to \forall\alpha < \beta + \omega^x A(\alpha))$.
In our partial setting we do not have in general

$$\frac{\Rightarrow A \qquad \Rightarrow A \to B}{\Rightarrow B}$$

Idea

Idea (Carlo): circumvent the MP argument step.
In UTS we can prove (by external induction) for all n that

$$\frac{Prog(A)}{\forall \alpha < \beta A(\alpha) \Rightarrow \forall \alpha < \beta + \omega^{\underline{n}} A(\alpha)}$$

Problem: How to use this fact?

Reflection on rules

Solution: Strengthening of reflection. Assume that Σ allows for the following derivation

$$\frac{\Gamma \Rightarrow \Delta}{\Theta \Rightarrow \Lambda}$$

Then a reflection on Σ should also include this fact

$$(R^*)\ \frac{Pr_\Sigma([\Gamma\dot{x}] \dot{\Rightarrow} [\Delta\dot{x}], [\Theta\dot{x}] \dot{\Rightarrow} [\Lambda\dot{x}]) \qquad \Gamma(x) \Rightarrow \Delta(x)}{\Theta(x) \Rightarrow \Lambda(x)}$$

$\mathrm{TIR}_{\mathcal{L}_T}(< \epsilon_0)$ in $R^*(\mathrm{UTS})$

Now we can formalize the external induction to get

$$\mathrm{Pr}_{\mathrm{UTS}}([\mathrm{Prog}(A)], [\forall\alpha < \beta A(\alpha)] \dot{\Rightarrow} [\forall\alpha < \beta + \omega^x A(\alpha)\dot{x}])$$

and with reflection we have

$$\frac{Prog(A)}{\forall\alpha < \beta A(\alpha) \Rightarrow \forall\alpha < \beta + \omega^x A(\alpha)}$$

Setting $\beta = 0$ we can then argue for $\mathrm{TIR}_{\mathcal{L}_T}(< \epsilon_0)$.

Open questions

- In classic theories we have a close connection between reflection and induction.
- Is it as close in partial logic?
- Is reflection able to close the (proof theoretic) gap between PKF and IKF?

Concluding remarks

- Theories of truth built on reflection principles are very well motivated.
- Reflection on simple truth sequents allows us to gain the compositional axioms of PKF.
- Reflection and induction are closely connected also in the partial setting.
- Reflection gives full induction.
- Reflection gives $\mathrm{TIR}_{\mathcal{L}_T}(<\epsilon_0)$.

Thank you!